An efficient unified spectral element Boussinesq model for a point absorber

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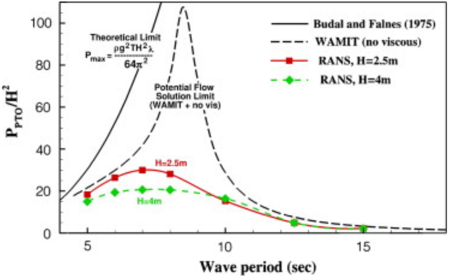
Wave energy converter

A point absorber is a floating structure which absorbs energy from all directions through its movements at/near the water surface. The typical diameter of point absorber is much smaller of the length of the waves in which it operates.

Why a depth averaged model?

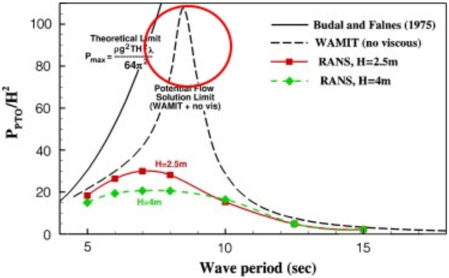
Linear model



Efficient and fast computation, not accurate in describing the high order nonlinear effects.

Why a depth averaged model?

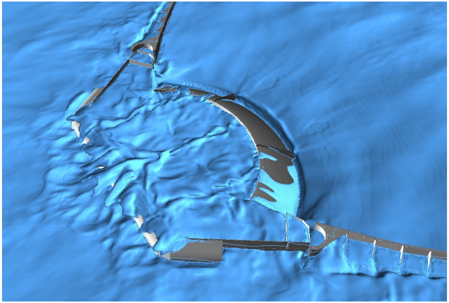
Linear model



Resonance peak presents and much higher than CFD simulation

Why a depth averaged model?

Reynolds averaged Navier Stokes model



RANS: high fidelity model but computationally impractical.

Why a depth averaged model?

Third Way

Depth averaged models

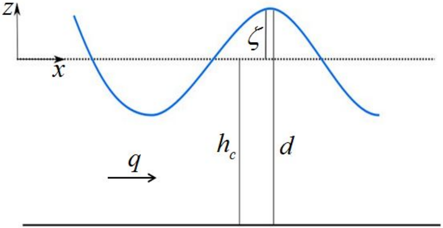
Depth averaged models

- Standard approach for nearly 20 years in the coastal engineering community
- Already used to model structure
- Application with floating bodies:
  - Jiang, T. *Ship waves in shallow water*, (2001).
  - Lannes, D. *On the dynamics of floating structures*, (2017).
  - Godlewski, Edwige, et al. *Congested shallow water model: roof modelling in free surface flow*, (2018).

Depth averaged models
NSW

The model approximate the irrotational Euler equation at the second order of nonlinearity:

$$\mu = \kappa h$$
$$\mu^2 \ll 1$$



where κ is the wavenumber and h is the still water depth.

Depth averaged models
NSW

In one dimension, as first approximation, we have the nonlinear shallow water (NSW) equation

$$\begin{cases} d_t + q_x = 0 \\ q_t + \left(\frac{q^2}{d}\right)_x + dP_x = 0 \end{cases}$$

in conservative variables: d is the wave elevation, q is flow field and $P = P_{hy} = gd$ the hydrostatic pressure.

Depth averaged models

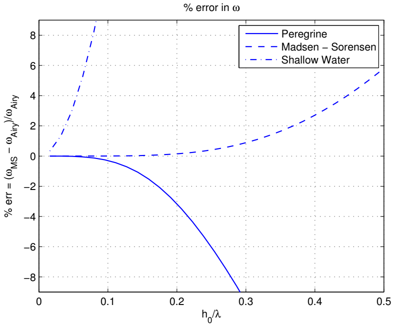
Madsen and Sørensen

Dispersion effects of order μ^2 can be added.

Weakly nonlinear correction: if $\epsilon = \frac{A}{h} \approx \mu^2$ with A the wave amplitude, we have for example the MS model:

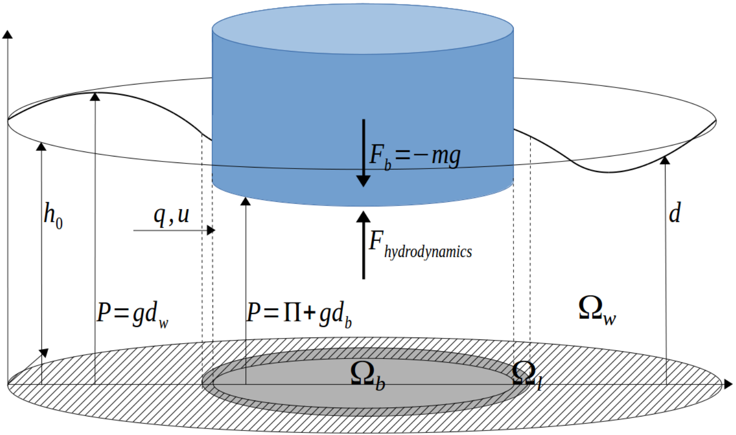
$$q_t - h^2 \beta q_{xxt} + \left(\frac{q^2}{d}\right)_x + dP_x - \alpha_{MS} h^2 dP_{xxx} = 0$$

The free parameters are be tuned to enhance the dispersion.



Depth averaged models

Domain setup



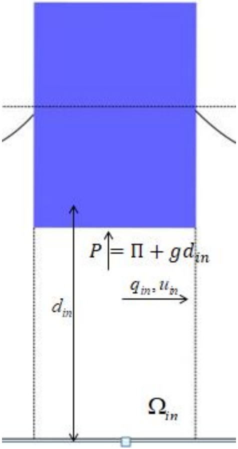
Depth averaged models

Inner Domain

Under the WEC, the motion is described by a NSW model

$$\begin{cases} d_t + q_x = 0 \\ q_t + \left(\frac{q^2}{d}\right)_x + dP_x = 0 \end{cases}$$

where d is the elevation of the body and $P = \Pi + gd$ the pressure field



Depth averaged models

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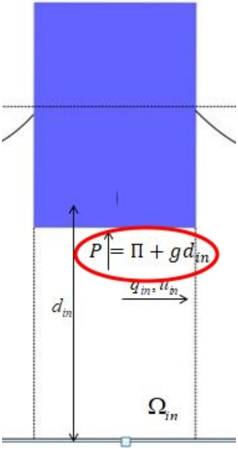


Depth averaged models

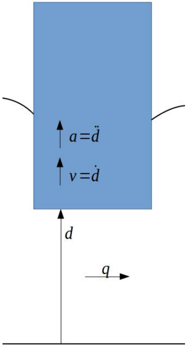
Pressure field

From the body mass equation we have

$$q_x = -d_t$$

$$q_{xt} = -d_{tt} = -a$$

a acceleration of the body



The acceleration can be evaluated from the balance of the forces $\mathcal{F}$ applied to the body

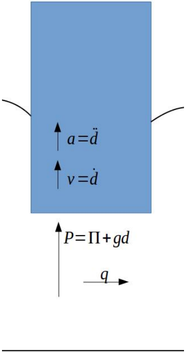
$$m_b a = -m_b g + \mathcal{F} = -m_b g + \rho_w \int_{\Omega_b} (P - gd)\partial S$$

Depth averaged models

The Inner pressure field can be evaluated taking the divergence of momentum equation

$$\partial_x \left[dP_x = q_t + \left(\frac{q^2}{d}\right)_x \right]$$

$$-(dP_x)_x = -a + \left(\frac{q^2}{d}\right)_{xx}$$



Depth averaged models

Final system

From the definition of the hydrostatic pressure, in the outer (free surface) domain we solve

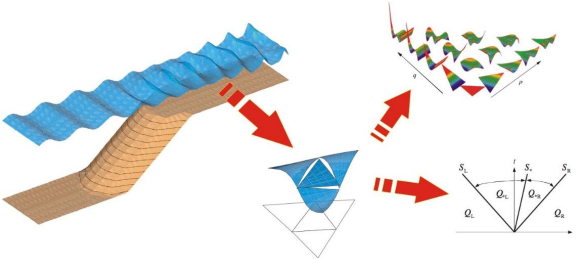
$$
\begin{aligned}
&P_t + g q_x = 0 \\
&q_t + \left(\frac{q^2}{d}\right)_x + dP_x = \Phi_{disp} \\
&\Phi_{disp} := h^2 \beta q_{xxt} + \alpha_{MS} h^2 d P_{xxx}
\end{aligned}
$$

and in the inner (under the body) domain

$$
\begin{aligned}
&-(dP_x)_x = -a + \left(\frac{q^2}{d}\right)_{xx} \\
&q_t + \left(\frac{q^2}{d}\right)_x + dP_x = 0 \\
&q_{xt} = -a, \quad P = \Pi + gd
\end{aligned}
$$

Spectral/*hp* Element method

- Higher (≥ 3) order convergence; more efficient for very long time integration
- potential *h*- and *p*-adaptivity



Spectral Element method

Discretization

The consistency of the system is kept at discrete leved adopting a first order derivative formulation. For the inner domain, it means

$$\left.\begin{aligned} -(dP_x)_x &= -a + \left(\frac{q^2}{d}\right)_{xx} \\ q_t + \left(\frac{q^2}{d}\right)_x + dP_x &= 0 \\ q_{xt} &= -a \\ P &= \Pi + gd \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &-(db)_x = -a + l_x \\ &q_t + l + b = 0 \\ &q_{xt} = -a \\ &P = \Pi + gd \end{aligned}\right.$$

Defining:

$$\left\{\begin{aligned} &b := P_x \\ &l := (q^2/d)_x \end{aligned}\right.$$

Spectral Element method

Discretization

Consider $b = P_x$:

- Evaluate the variational form
- Double integrate by part

$$\int_{\Omega} \varphi b dx = \int_{\Omega} \varphi P_x dx + \int_{\partial\Omega} \varphi(\hat{P} - P)\hat{n} dx$$

- Discretize the variables

We need only a derivative matrix operator and a projection matrix to solve the equation.

Spectral Element method

Discretization

At discrete level

$$\boldsymbol{M} b = \boldsymbol{D} P + \boldsymbol{C} \left(\hat{P} - P \right)$$

Depending on the penalty terms, we can collect the matrices as $\boldsymbol{Q} = \boldsymbol{D} + \boldsymbol{C}$

Spectral Element method

Discretization

At discrete level

$$\boldsymbol{M}b = \boldsymbol{D}P + \boldsymbol{C}\left(\hat{P} - P\right)$$

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REMARK

- The precision of the model depends on the choice of $\boldsymbol{D}$ and the basis function φ
- The stability depends on the choice of the penalty terms

Spectral Element method

Discretization

Combining the first derivative matrices, the discrete inner domain system becomes

$$\begin{cases} \boldsymbol{Q} q_t = -\boldsymbol{M} a \\ \boldsymbol{M} q_t + \boldsymbol{Q}(q^2/d) + \boldsymbol{\mathcal{D}}_d \boldsymbol{Q} P = 0 \\ -\boldsymbol{Q}\boldsymbol{M}^{-1}\boldsymbol{\mathcal{D}}_d \boldsymbol{Q} P = -\boldsymbol{Q} a + \boldsymbol{Q}\boldsymbol{M}^{-1}\mathcal{D}_d \boldsymbol{Q}(q^2/d) \end{cases}$$

and the outer domain system

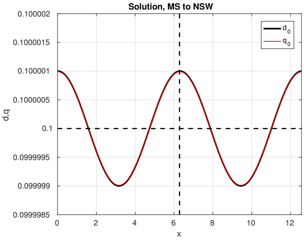
$$\begin{cases} \boldsymbol{M} d_t + \boldsymbol{Q} q = 0 \\ \boldsymbol{M} q_t + \boldsymbol{Q}(q^2/d) + \boldsymbol{\mathcal{D}}_d \boldsymbol{Q} P = \boldsymbol{\Phi}_{disp} \end{cases}$$

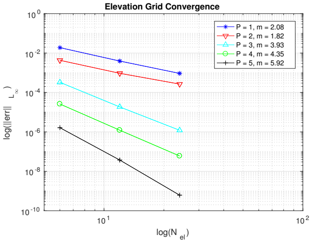
Convergence

Wave propagation and manufactured solution

The first tests were done propagating a wave through different domains and check the convergence using a manufactured solutions:

$$d(x,t) = f_1(x - ct), \quad q(x,t) = f_2(x - ct)$$

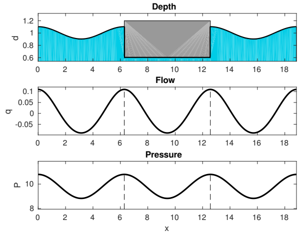


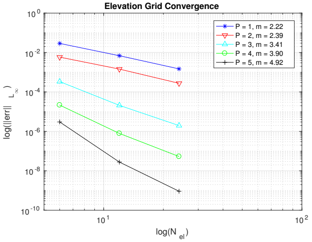


Convergence

Wave-fixed box convergence

Second we tested the convergence of the model with a fixed box in the center. Using a manufactured solution:

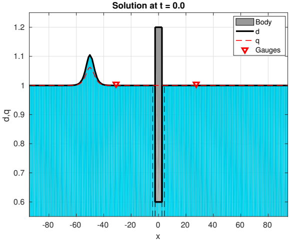




Wave - Pontoon Coupling

Fixed Box - Benchmark

We reproduced the benchmark in *Rodriguez and Spinneken (2016)* of a fixed pontoon interacting with a solitary wave.



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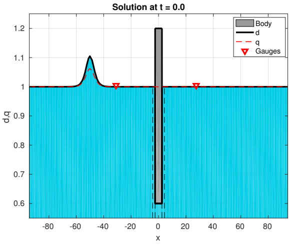

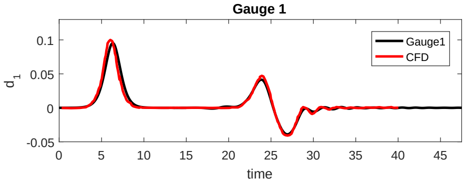

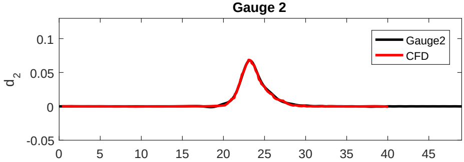

Forced and Decay test

Forced motion test

The forced motion reproduced the solution of *D. Lannes (2016)*. The figure shows the evolution of the wetting point for the forced motion test and the exact solution.

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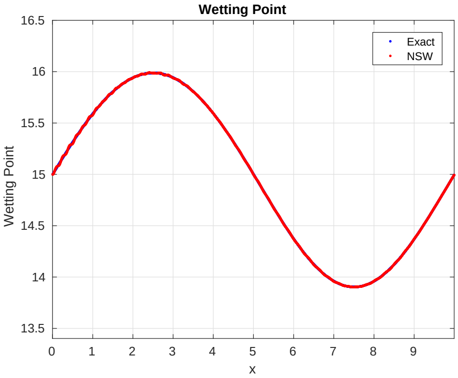

Forced and Decay test

Decay test

The decay test reproduced the numerical solution of *D. Lannes (2017)*. The figure shows the evolution of the center of gravity for the decay test and the numerical solution.

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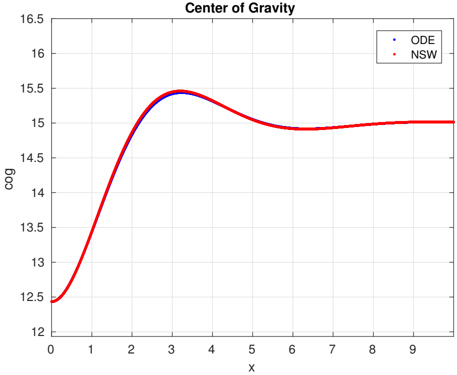
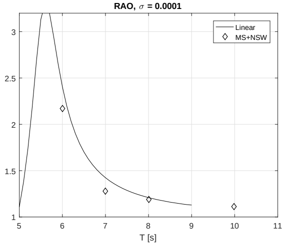


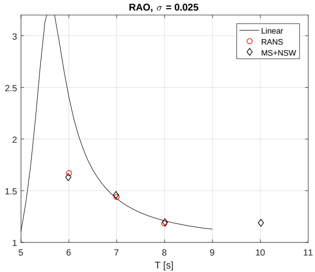
Heading Body

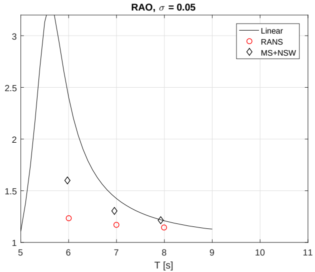
Single Body

The free heaving motion is tested with a rectangular box interacting with waves of different steepness and period. The *Response amplitude operator* (RAO) of the Boussinesq case is tested against a linear code and a CFD one.



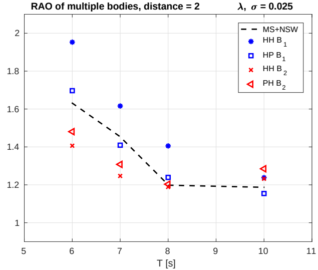
Heaving Body





Ongoing work
Multiple Bodies

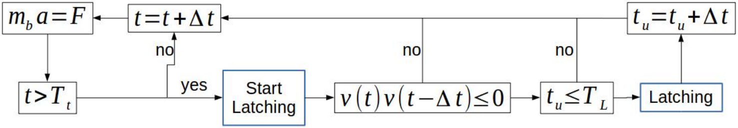
Our routine permits us to simulate easily multiple bodies:



The RAO is evaluated for both bodies in case are both heaving boxes (HH in the legend) or one of them is a pontoon (HP or PH).

Ongoing work
Latching

Control technique to improve body response.
The body is held still (latched) in some time windows to amplify the oscillations once released.

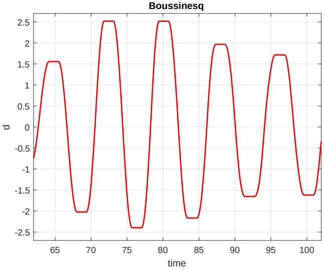


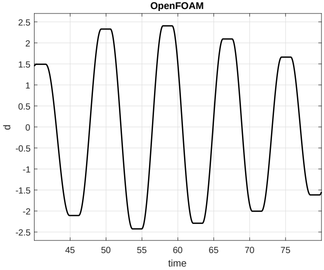
The velocity of the body is set to zero for a time T_L once the body reaches its peak and then is freed to continue the movement.

Ongoing work

Latching

Latching test with a box interacting with a wave of period $T = 8s$ and steepness $\sigma = 0.025$





Follow up

- Improved penalty terms and numerical stabilization
- Case test with latching and PTO
- 2D/3D model

Acceleration

Discrete acceleration equation

$$m_b a = -m_b g + \rho_w \boldsymbol{w}^T (P - gd)$$

where $\boldsymbol{w}$ are the Gauss-Lobatto-Legendre weights.

Acceleration

Substituting the expression of the pressure the acceleration can be evaluated as

$$(m_b + \mathcal{M}_{add})\, a = -m_b g$$
$$-\, \boldsymbol{w}^T \left(\left(\boldsymbol{Q}\boldsymbol{M}^{-1}\boldsymbol{\mathcal{D}}_d\boldsymbol{Q} \right)^{-1} (\boldsymbol{Q}\boldsymbol{M}^{-1}\boldsymbol{Q})(q^2/d) + g\boldsymbol{M}d \right)$$

where we have defined the added mass $\mathcal{M}_{add}$

$$\mathcal{M}_{add} = -\boldsymbol{w}^T \left(\boldsymbol{Q}\boldsymbol{M}^{-1}\boldsymbol{\mathcal{D}}_d\boldsymbol{Q} \right)^{-1} \boldsymbol{w}.$$

It can be shown that $\mathcal{M}_{add} \geq 0$ provided that $\boldsymbol{Q}\boldsymbol{M}^{-1}\boldsymbol{\mathcal{D}}_d\boldsymbol{Q}$ is invertible.